Fractal structures and multiparticle effects in soliton scattering

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We study in detail the interaction of composite solitary waves and consider, as an example, the breather collisions in a weakly discrete sine-Gordon equation. We reveal a physical mechanism of fractal soliton scattering associated with multiparticle effects, and demonstrate chaotic interaction of two breathers with incommensurable frequencies.

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The rapidly growing interest in the study of the solitary-wave interaction in nonintegrable nonlinear models [1–3] is explained by the possibility of observing many of the predicted effects experimentally [4], including the soliton energy and momentum exchange. One of the most intriguing properties of soliton interactions in nonintegrable models is the observation of the fractal nature of their scattering, first discussed for kink-antikink collisions in the so-called ϕ^4 model [5]. The main features of fractal soliton scattering are usually explained by the excitation of the soliton internal mode, which is an important property of solitary waves of many nonintegrable soliton-bearing models [6]. Thus, the physics of fractal soliton scattering can be understood as a resonant energy exchange between the soliton translational motion and its internal mode [7]. A similar mechanism was revealed for the interaction of a kink with a localized impurity [8].

In this paper, we discuss a different physical mechanism of fractal soliton scattering and consider, for simplicity and historical tradition, the well-known model described by the sine-Gordon (SG) equation weakly perturbed by discreteness effects. In this case, the role of the soliton internal mode is negligible [6], and the fractal structures observed in soliton scattering should be explained by a qualitatively different mechanism. In particular, we study the breather scattering in such a model, and describe several interesting phenomena that can be understood as manifestations of multiparticle effects in the soliton collisions, due to resonant coupling between the ''atomic'' and ''molecular'' degrees of freedom of the colliding composite solitons.

We consider a discrete version of the well-known SG equation [often called the Frenkel-Kontorova (FK) model [9]] that describes the dynamics of a one-dimensional atomic chain with the Hamiltonian

$$H=\sum_n \left[\frac{1}{2}\dot{u}_n^2+\frac{1}{2h^2}(u_{n+1}-u_n)^2+(1-\cos u_n)\right],$$

where h is the lattice spacing and u_n is the displacement of the nth particle, so that the first term is the kinetic energy of the nth particle, the second term describes the energy of the elastic coupling between the nth and $(n+1)$th particles, and the last term is the energy of the nth particle in a periodic on-site potential. The corresponding equations of motion are

$$\ddot{u}_n-h^{-2}(u_{n-1}-2u_n+u_{n+1})+\sin u_n=0, \tag{1}$$

and, in the long-wavelength approximation, they reduce to the well-known SG equation $u_{tt}-u_{xx}+\sin u=0$, which is completely integrable by the inverse scattering transform and, therefore, does not possess any kind of many-particle effect in the soliton interactions [10]. However, such many-soliton effects may appear for the interaction of more than two solitary waves in the presence of small perturbations (see, e.g., Refs. [10,11]). In our case, a perturbation is generated by a weak lattice discreteness (h is small compare to the kink width, which is equal to 1), and it can be taken into account by a small dispersive correction to the SG equation in the form $(h^2/12)u_{xxxx}$. Physically, the condition of three-particle inelastic collisions requires that all three solitons meet at one point [10,11], an event that is difficult to accomplish. However, if the colliding solitons are composite, i.e., they consist of several components or they present a bound state of two simpler solitary waves, the many-particle effects can already be observed for collisions of two composite solitons. The simplest known example of this kind is the kink-breather interaction in the SG model [12], where energy exchange between colliding solitons is observed even in the weakly discrete limit. Thus, the energy exchange and multiparticle effects seem to be common features of a cold gas of kinks and breathers, where all solitary waves have nearly equal velocities and their collision is accompanied by a strong energy and momentum exchange with a small amount of radiation emitted.

Multisoliton collisions in the SG equation. To study the breather scattering in a weakly discrete SG equation, first we start from the exact solution that describes the interaction of four kinks. Such a four-soliton solution of the SG equation can be obtained by means of the Bäcklund transformation [13],

$$u(x,t)=4\tan^{-1}[f(x,t)/g(x,t)], \tag{2}$$

where

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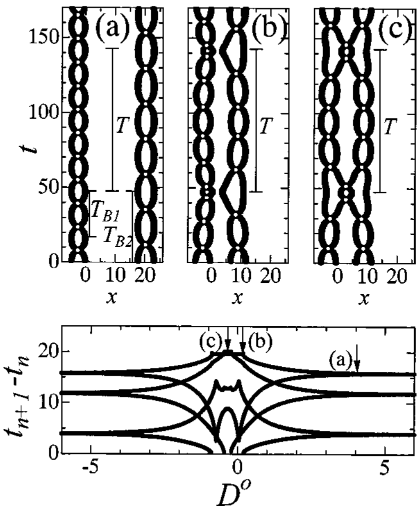

FIG. 1. Top: Dynamics of the breather interaction in the unperturbed SG equation: (a) $D^0=4.0$, (b) $D^0=0.2$, and (c) $D^0=-0.34$. Bottom: The difference $t_{n+1}-t_n$ as a function of D^0, where t_n is the time of the nth peak of the kinetic energy.

$$f(x,t)=\sum_{m=0}^{1}\sum_{C_{2m+1}^{4}} b_{k_1,\ldots,k_l}\exp(\eta_{k_1}+\cdots+\eta_{k_l}), \quad (3)$$

$$g(x,t)=\sum_{m=0}^{2}\sum_{C_{2m}^{4}} b_{k_1,\ldots,k_l}\exp(\eta_{k_1}+\cdots+\eta_{k_l}), \quad (4)$$

and C_j^4 stands for a sum over all combinations of j elements from four elements. In Eqs. (3) and (4), the soliton parameters are defined as $\eta_i=[(1+\xi_i^2)/2\xi_i](x-x_i^0)-[(1-\xi_i^2)/2\xi_i]t$, where ξ_i is a complex eigenvalue, and x_i^0 is the initial position of the ith soliton. The coefficients $b_{k_1,\ldots,k_l}=\Pi_{m<n}^{l}a_{k_m k_n}$ for $l\geqslant 2$, and $b_{k_1,\ldots,k_l}=1$ for $l=0,1$, where $a_{ij}=-(\xi_i-\xi_j)^2/(\xi_i+\xi_j)^2$. A real ξ_i corresponds to a kink ($\xi_i>0$) or an antikink ($\xi_i<0$) moving with the velocity $V_K=(1-\xi_i^2)/(1+\xi_i^2)$.

Any two kinks, say i and j, can be coupled into a breather. In this case, we take $x_i^0=x_j^0=x_B^0$ and the parameters ξ_i and ξ_j are complex conjugate, $\xi_i=\xi_j^*=\alpha_{ij}+i\beta_{ij}$. The breather velocity V_B, its frequency ω_B, and the period of oscillation T_B are defined as $V_B=(1-R_{ij}^2)/(1+R_{ij}^2)$, $\omega_B=2\beta_{ij}/(1+R_{ij}^2)$, and $T_B=2\pi/\omega_B$, where $R_{ij}^2=\xi_i\xi_j=\alpha_{ij}^2+\beta_{ij}^2$.

A four-soliton solution of the SG equation is a nonlinear superposition of N_K kinks and N_B breathers, $N_K+2N_B=4$, so that the total soliton energy is equal to the sum of the energies of the kinks and, in this sense, multiparticle effects are absent. However, some reminiscence of multisoliton effects can be observed even in the unperturbed SG equation when a pair of breathers collide with velocities very close to each other. To see that, for the breather-breather (BB) solution we take the parameters $\xi_1=\xi_2^*$, $\xi_3=\xi_4^*$ and $x_1^0=x_2^0=x_{B1}^0$, $x_3^0=x_4^0=x_{B2}^0$. In Figs. 1(a)–1(c), we show the time evolution of the unperturbed BB system with the initial velocities $V_{B1}=V_{B2}=0$ and frequencies $\omega_{B1}=0.2$ and $\omega_{B2}=(2/3)\omega_{B1}$, and we vary the initial distance between breathers, $D^0=x_{B2}^0-x_{B1}^0$. The bold curves in Figs. 1(a)–1(c) show the regions with energy density greater than 3.0 and thus the breather constituents (kinks and antikinks) can be easily seen. In Fig. 1(a), the breathers are far from each other and therefore they do not interact, but in Figs. 1(b) and 1(c) they overlap significantly, and we can observe their interaction as periodic modulations of the unperturbed kink trajectories.

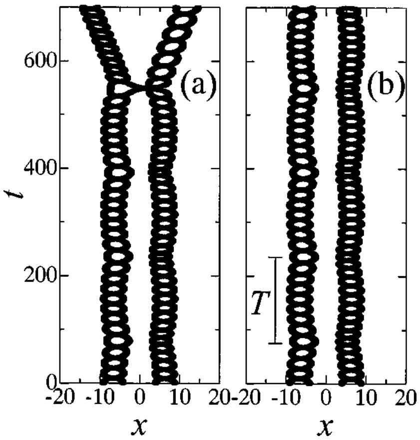

FIG. 2. Comparison of the breather interactions in (a) the discrete model (1) and (b) the unperturbed SG equation, for $D^0=-0.96$, $V_{B1}=V_{B2}=0$, $\omega_{B1}=0.2$, and $\omega_{B2}=(4/5)\omega_{B1}$.

In Figs. 1(a)–1(c), the collisions of two subkinks that constitute a breather are marked by open circles. When two or more subkinks collide at one point, a sharp peak of the kinetic energy is observed. We denote the time of the nth peak by t_n and plot the value $t_{n+1}-t_n$ as a function of the initial distance between the breathers (see Fig. 1, bottom). When the breathers are well separated, the system has three characteristic periods. For the interacting breathers, the number of periods is doubled, and for the strongly overlapping breathers, it can be equal to four or five. The period $T=3T_{B1}=2T_{B2}$ remains the same in all cases shown.

Thus, two overlapping breathers with commensurable frequencies $M\omega_{B1}=N\omega_{B2}$ can be regarded as a four-kink composite state with the period $T=NT_{B1}=MT_{B2}$. The period T does not depend on the distance D^0 but the appearance of the sharp peaks in the kinetic energy within this period depends on D^0. One can see that, if the ratio M/N is an irreducible fraction with large M or N, then the period T is also large. For irrational M/N, T is infinitely large, so that the breather motion looks aperiodic, and the peaks of the kinetic energy also appear aperiodically. In this sense, we can say that even the unperturbed SG equation contains ‘‘a seed of chaos.’’

Breather collisions in a discrete model. In order to understand how the properties of the SG model are broken by nonintegrability, we study the dynamics of BB collisions in a weakly discrete SG model ($h=0.2$). Equations (1) were integrated numerically with the use of Störmer method of order 6 with the time step $\Delta t=10^{-4}$. The numerical data reported below do not vary essentially with further decrease of Δt. To set the initial conditions, we use the analytical BB solution of the unperturbed SG equation.

In Fig. 2 we compare the BB interactions in the discrete FK system with the exact BB solution of the unperturbed SG

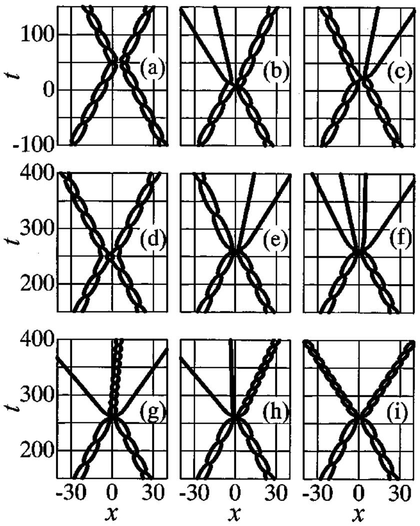

FIG. 3. Examples of the breather scattering in a perturbed SG model for the initial velocities $V_{B1}=-0.2$ and $V_{B2}=0.2$, and the frequencies $\omega_{B1}=0.1$ and $\omega_{B2}=(4/5)\omega_{B1}$. (a) $D^0=20.0$, (b) $D^0=4.3$, (c) $D^0=8.1$, (d) $D^0=100.0$, (e) $D^0=104.1$, (f) $D^0=104.13$, (g) $D^0=104.26$, (h) $D^0=104.285$, and (i) $D^0=104.33$.

equation with zero initial velocities of both breathers. The SG equation predicts an oscillatory motion of the breathers, as demonstrated in Fig. 2(b), because they attract each other, oscillating nearly in phase, and then repel each other, oscillating nearly out of phase. The period of oscillations is $T=5T_{B1}=4T_{B2}$, where T_{B1} and T_{B2} are the periods of the individual breathers.

In a weakly discrete SG system [see Fig. 2(a)], the attraction between the colliding breathers when they are in phase is not fully compensated by the repulsion when they are out of phase. As a result, the mean distance between the breathers becomes smaller and they collide. The weak attraction of the breathers, which is due to the fact that the attraction force is not fully compensated by the repulsion force, appears because of the model discreteness, which breaks its integrability. Another nontrivial effect of discreteness is the energy and momentum exchange between the colliding breathers. During the interaction, the breathers can gain some velocity and the BB system can split into two independent breathers or even into individual subkinks.

Criteria for inelastic collisions. The inelasticity of many-soliton collisions in a weakly discrete model drastically increases in the vicinity of a separatrix solution of the unperturbed SG equation. As a matter of fact, we notice a simple practical criterion for such inelasticity based on the fact that in a weakly discrete system the effective perturbation plays a key role. Indeed, our numerical results suggest that the BB collisions in the discrete system are strongly inelastic if and only if the term in the form of the derivative u_{xxxx} calculated for the unperturbed SG solution has a pronounced maximum.

Typical collisions of two breathers are shown in Figs. 3(a)–3(i). In Figs. 3(a) and 3(d), the collisions are practically elastic, i.e., there is no energy and momentum exchange between breathers. In all other cases, the BB collisions are inelastic. In Fig. 3(a), at the moment of collision, the breathers oscillate nearly out of phase, and that is why they repel each other. In Fig. 3(d), the breathers oscillate nearly in phase. Nevertheless, the collision in Fig. 3(d) is elastic because it occurs without involving three- or four-kink collisions. In contrast, in Figs. 3(b) and 3(c), three-kink collisions can be seen and, in Figs. 3(e)–3(i), all four kinks manifest themselves in the collision. Additionally, one of the colliding breathers (or even both of them) can break up into subkinks. The breaking up takes place only for breathers with sufficiently small frequencies; otherwise, the inelasticity of the collision causes energy and momentum exchange between the breathers, as is shown in Fig. 3(i).

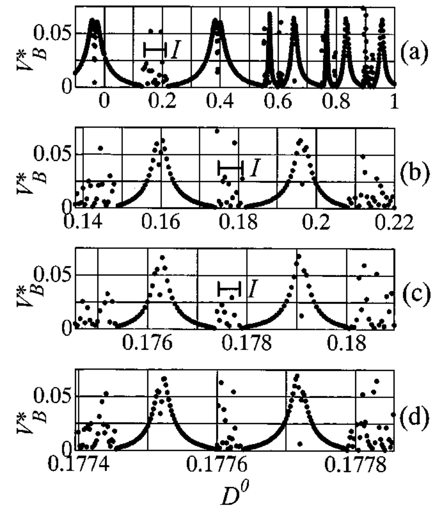

FIG. 4. Fractal structure (four scales are shown) of breather collisions shown as the function $V_B^*(D^0)$.

Inelastic collisions occur only when more than two subkinks collide at one point. This event has a small probability and thus the multiparticle effects are observed in a narrow region of D^0. An important exception is the case when the breathers collide with very small relative velocities. In this case, the time of collision is much larger than the period of the breather oscillation and, regardless of the value of D^0, the BB collisions are always inelastic. We also mention that the radiation losses are very small in all the cases discussed.

Fractal structures in breather scattering. We study in more detail the process of the breakup of the BB system into two separate breathers. A typical example of such a breakup process is shown in Fig. 2(a). We select the parameters $\omega_{B1}=0.2$, $\omega_{B2}=(4/5)\omega_{B1}$, $V_{B1}=V_{B2}=0$, and change the initial distance between the breathers. After breaking up, the two independent breathers move in opposite directions. The absolute values of their velocities are nearly equal because in a weakly discrete chain momentum conservation is nearly fulfilled and, for our choice of parameters, the breathers have nearly the same energies.

We denote the breather velocity after splitting as V_B^*, and plot this value in Fig. 4(a) as a function of the initial distance D^0. The function $V_B^*(D^0)$ shows the property of self-similarity at different scales usually associated with fractal scattering. Four levels of such similarity are presented in Figs. 4(a)–4(d), where each succeeding figure is shown for the interval expanded from a smaller region marked by I in the preceding one. The expansion coefficient is about 13.5

for each step. At each scale, the function $V_B^*(D^0)$ looks like an alternation of smooth and chaotic domains. However, at larger magnification, each chaotic domain again contains chaotic regions and smooth peaks. Thus, the output velocity $V_B^*(D^0)$ is actually a set of smooth peaks of different scales. In some regions, the width of the peaks vanishes and the density of the peaks goes to infinity. At the same time, the height of the peaks remains the same at each scale. The fractal structure of the function $V_B^*(D^0)$ proves the chaotic character of breather scattering in a weakly discrete model.

The fractal nature of breather collisions can have a simple physical explanation. As was shown above, in a weakly discrete (and, therefore, weakly perturbed) system the breathers attract each other with a weak force. As can be seen from Fig. 4, the chaotic regions appear where the extrapolation of the smooth peaks gives nearly zero velocity V_B^*. In these regions, the breathers gain a very small velocity after interaction and subsequent splitting. With such a small initial velocity, the breathers cannot overcome their mutual attraction and collide again. In the second collision, due to momentum exchange, the breathers can acquire an amount of kinetic energy sufficient to escape each other, but there exists a finite probability of gaining kinetic energy below the escape limit. In the latter case, the breathers will collide for a third time, and so on. Thus, a series of collisions leads to a resonant energy exchange between the ''atomic'' (kink's translational) and ''molecular'' (relative oscillatory) breather degrees of freedom, and to fractal scattering.

Above, we have analyzed the collision of breathers with commensurable frequencies and the results presented in Fig. 3 look similar for other cases. In particular, the collision is extremely sensitive to the mutual phase of the colliding breathers, which depends on D^0. Obviously, if the breathers have incommensurable frequencies, the mutual phase becomes an aperiodic function of D^0, which results in chaotic breather scattering.

In conclusion, we have revealed a physical mechanism of fractal soliton scattering that is not associated with the kink's internal modes. This mechanism of inelastic soliton scattering has been demonstrated for the simplest case of breather (a bound state of a kink and antikink) scattering in the sine-Gordon model, but it seems to be a common feature of many nonintegrable nonlinear models that support composite solitary waves, e.g., the models recently analyzed in Refs. [1].

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[1] A.V. Buryak *et al.*, Phys. Rev. Lett. **82**, 81 (1999); E.A. Ostrovskaya *et al.*, *ibid.* **83**, 296 (1999); Z.H. Musslimani *et al.*, *ibid.* **86**, 799 (2001); J.J. García-Ripoll *et al.*, e-print nlin.PS/0006047.
[2] C. Anastassiou *et al.*, Phys. Rev. Lett. **83**, 2332 (1999).
[3] J. Yang and Yu Tan, Phys. Rev. Lett. **85**, 3624 (2000).
[4] G.I. Stegeman and M. Segev, Science **286**, 1518 (1999).
[5] P. Anninos, S. Oliveira, and R.A. Matzner, Phys. Rev. D **44**, 1147 (1991).
[6] Yu.S. Kivshar *et al.*, Phys. Rev. Lett. **80**, 5032 (1998).
[7] D.K. Campbell, J.F. Schonfeld, and C.A. Wingate, Physica D **9**, 1 (1983).
[8] Yu.S. Kivshar, F. Zhang, and L. Vázquez, Phys. Rev. Lett. **67**, 1177 (1991).
[9] See, e.g., O.M. Braun and Yu.S. Kivshar, Phys. Rep. **306**, 1 (1998).
[10] Yu.S. Kivshar and B.A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[11] H. Frauenkron, Yu.S. Kivshar, and B.A. Malomed, Phys. Rev. E **54**, 2244 (1996).
[12] S.V. Dmitriev *et al.*, Phys. Lett. A **246**, 129 (1998); Phys. Rev. E **61**, 5880 (2000); A.E. Miroshnichenko *et al.*, Nonlinearity **13**, 837 (2000).
[13] See, e.g., R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, and H.C. Morries, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982).